On the Improbability of a Random Distribution of the Stars in Space.

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(1) The object of the present paper is to discuss certain statistical distributions of the stars, and to consider how far they are in accordance with our experience so far as that extends. For direct tests we require far more numerous determinations of parallax than are at present available.

The largest sample of parallaxes hitherto published is that due to Kapteyn,* but, for reasons elsewhere stated, I do not feel able to include this in my survey. Small series collected by Newcomb and determined by the Yale observers will be considered below.

The main point of my investigation will, however, turn on certain parallax constants, which frequently arise in stellar statistical investigations, and which are capable of determination without a knowledge of absolute parallax.

I shall use the following notation: m = a stellar magnitude, D = an absolute stellar distance, $\pi = c/\mathrm{D}$ = a stellar parallax,† c being a constant, depending on the unit of measurement and the diameter of the earth's orbit. $\mathrm{P} = \log_{10}\pi = \log_{10}c - \log_{10}\mathrm{D}$, is the logarithm of the parallax. A rule over any of these quantities will denote their mean values. σ_m, σ_D, σ , σ_P will represent their standard deviations, or square roots of mean square deviations from the mean values.

V will denote what is termed a coefficient of variation, or the ratio of a standard deviation to the corresponding mean value.

$\mathrm{V_D} = \sigma_\mathrm{D}/\bar{\mathrm{D}}$, $\mathrm{V}_\pi = \sigma_\pi/\bar{\pi}$, and σ_P are the parallax constants which naturally arise in the course of stellar statistics, and which to some extent can be measured without the aid of absolute parallax determinations.

The letter r will be used throughout for a correlation coefficient, the attached subscripts denoting the characteristic whose association is under discussion.

(2) Let L_1 represent the light of a star; then this light depends: (a) on the size of the star; (b) its intrinsic brightness; (c) its distance from us; and,

* "The Parallaxes of 3650 Stars of different Galactic Latitudes," J. C. Kapteyn and W. de Sitter, 'Publications of the Astronomical Laboratory at Groningen,' No. 20, 1908.

† I have retained π for parallax in accordance with much astronomical usage, although it introduces some confusion when $\pi = 3{\cdot}14\ldots$ occurs in the same equations, and still more if it appears in the form $\sin\pi$. p and P are needed for many other purposes. In manuscript I invariably write Π, but this never meets with the printer's approval.

possibly (*d*), on the existence of interstellar absorption. Or, if λ represent an absorption coefficient, we shall have

$$L_1 = \frac{F_1}{D_1^2} 10^{-\lambda D_1},$$

or, $\log_{10} L_1 = \log_{10} F_1 - 2 \log_{10} D_1 - \lambda D_1$, where F_1 depends solely on the constitution of the star.

But if we measure light in terms of the stellace, *i.e.*, the light, L_0, of a star of tenth-magnitude as unit, and use Pogson's value of the constant, we have

$$\log_{10} L_1 = 4 - 0{\cdot}4\, m_1 + \log_{10} L_0,$$

and transferring to parallax,

$$4 - 0{\cdot}4\, m_1 = \log_{10} F_1 - \log_{10} L_0 - 2 \log_{10} c + 2 \log_{10} \pi - \lambda c/\pi,$$

or

$$m_1 = f_1 - 5 \log_{10} \pi + 2{\cdot}5\, \lambda c/\pi.$$

Here f_1 is entirely independent of the star's parallax; it contains a constant, $10 + 2{\cdot}5 \log_{10} L_0 + 5 \log_{10} c$, which disappears when we consider deviations from mean values, and a quantity, $2{\cdot}5 \log_{10} F_1$, which depends solely on the intrinsic brightness and size of the star. With our present knowledge there is no reason whatever for supposing any association between the distance of a star from the sun and its size or intrinsic brightness. We shall therefore suppose $r_{\pi f} = 0$. Further, if there be an absorption term, we may suppose it so small that for our first approximation we can put $\lambda = 0$. Thus we write:

$$m_1 = f_1 - 5 P_1,$$

whence it follows that for any series of stars

$$\bar{m}_1 = \bar{f}_1 - 5 \bar{P}_1,$$

$$\sigma_{m_1}^2 = \sigma_{f_1}^2 + 25 \sigma_{P_1}^2.$$

It is accordingly clear that for any system of stars, the easily calculable σ_{m_1} must be greater than the value $5\sigma_{P_1}$ obtained from any suggested theoretical distribution. We thus have a means of testing the possible non-applicability of various theoretical systems.

Again, let there be a second magnitude, m_2, arising from a light phenomenon at the same or sensibly the same distance from us as m_1. This can arise in the cases of variable stars and of double stars. Then we have

$$m_2 = f_2 - 5 P_1,$$

$$\bar{m}_2 = \bar{f}_2 - 5 \bar{P}_1,$$

$$\sigma_{m_2}^2 = \sigma_{f_2}^2 + 25 \sigma_{P_1}^2.$$

Whence, if N be the number of stars dealt with,

$$\frac{S(m_1 - \bar{m}_1)(m_2 - \bar{m}_2)}{N} = \frac{S(f_1 - \bar{f}_1)(f_2 - \bar{f}_2)}{N} + 25 \frac{S(P - \bar{P})^2}{N},$$

i.e.,

$$\sigma_{m_1}\sigma_{m_2} r_{m_1 m_2} = \sigma_{f_1}\sigma_{f_2} r_{f_1 f_2} + 25 \sigma_P^2.$$

Thus, if there be no association between the actual lights emitted, we should have

$$\sigma_{m_1}\sigma_{m_2}r_{m_1m_2} = 25\sigma_{\rm P}^2,$$

the left-hand side of which is determinable for any series by observation and calculation. This provides another method of ascertaining whether a theoretical distribution is applicable in the case of light phenomena at the same distance which are assumed to be independent.

In general

$$r_{f_1f_2} = \frac{\sigma_{m_1}\sigma_{m_2}r_{m_1m_2}-25\sigma_{\rm P}^2}{\sqrt{\sigma_{m_1}^2-25\sigma_{\rm P}^2}\sqrt{\sigma_{m_2}^2-25\sigma_{\rm P}^2}},$$

which enables us to ascertain the actual relationship between light-giving capacities, if we can hypothesise a reasonable distribution for the stars under discussion. Thus, for example, we might, perhaps, assume to a first approximation that double stars are scattered at random in space, and might inquire whether any relationship exists between the intrinsic brilliancies of the primary and secondary.

(3) My proposed discussion will naturally break up into two parts. First, the consideration of what values these parallax constants take for theoretical stellar distributions; and secondly, how far any system of stars observable by us, and for that very reason a selected system, is likely to be a random sample of such theoretical systems.

I consider first some sample theoretical distributions.

(4) Both Newcomb and Pickering have investigated in the first place the effect on parallax frequencies of the uniform distribution of stars through space. This appears an obviously reasonable suggestion, although, as we advance, weighty reasons will arise for modifying this standpoint. I take first:—

The Values of $V_{\rm D}$, V_π, *and* $\sigma_{\rm P}$ *for a Uniform Distribution of the Stars throughout a Sphere of radius b.*

Let ρ be the density of the stars per unit volume, then

$$\tfrac{4}{3}\pi b^3\rho\bar{\rm D} = \int_0^b 4\pi r^2\rho\, r dr, \quad \text{or} \quad \bar{\rm D} = \tfrac{3}{4}b,$$

$$\tfrac{4}{3}\pi b^3\rho\sigma_{\rm D}^2 = \int_0^b 4\pi r^2\rho\,(r-\tfrac{3}{4}b)^2\, dr, \quad \text{or} \quad \sigma_{\rm D}^2 = \tfrac{3}{80}b^2.$$

Hence $$V_{\rm D} = \sqrt{\tfrac{1}{15}} = 0{\cdot}258.$$

Again

$$\tfrac{4}{3}\pi b^3\rho\bar{\varpi} = \int_0^b 4\pi r^2\rho\frac{c}{r}\, dr, \quad \text{or} \quad \bar{\varpi} = \frac{3}{2}\frac{c}{b},$$

$$\tfrac{4}{3}\pi b^3\rho\sigma_\pi^2 = \int_0^b 4\pi r^2\rho\left(\frac{c}{r}-\frac{3}{2}\frac{c}{b}\right)^2 dr, \quad \text{or} \quad \sigma_\pi^2 = \frac{3}{4}\left(\frac{c}{b}\right)^2.$$

Hence $$V_\pi = \sqrt{\tfrac{1}{3}} = 0{\cdot}577.$$

Lastly, if μ be the modulus $= 0{\cdot}43429$,

$$\tfrac{4}{3}\pi b^3 \rho \bar{P} = \int_0^b 4\pi r^2 \rho\mu \log_e \frac{c}{r}\, dr,$$

and, integrating by parts, $$= \tfrac{4}{3}\pi b^3 \mu \log_e \frac{c}{b} + \frac{4\pi b^3}{9}\mu,$$

or $$\bar{P} = \mu\left(\log_e \frac{c}{b} + \frac{1}{3}\right).$$

Again $$P - \bar{P} = \mu\left(\log_e \frac{c}{r} - \log_e \frac{c}{b} - \frac{1}{3}\right)$$
$$= \mu\,(k - \log_e r),$$

where $k = \log_e b - \tfrac{1}{3}$. Thus we have

$$\tfrac{4}{3}\pi b^3 \rho\, \sigma_P^2 = \int_0^b 4\pi r^2 \rho \mu^2 (k - \log_e r)^2\, dr = \int_0^b 4\pi\mu_2 (\log_e r)^2\, d\left(\tfrac{1}{3}r^3\right) - \mu^2 k^2 \frac{4\pi b^3}{3}.$$

Hence, integrating twice by parts and evaluating the indeterminate terms, we find

$$\tfrac{4}{3}\pi b^3 \rho\, \sigma_P^2 = \tfrac{4}{3}\pi b^3 \mu^2 \left[(\log_e b)^2 - \tfrac{2}{3}\log_e b + \tfrac{2}{9} - k^2\right],$$

or $$\sigma_P^2 = \tfrac{1}{9}\mu^2, \qquad \sigma_P = \tfrac{1}{3}\mu = 0{\cdot}1448.$$

The special quantity generally needed, $25\sigma_P^2$, is given by

$$25\sigma_P^2 = 2{\cdot}778\,\mu^2 = 0{\cdot}5240.$$

Now the results reached are, supposing the stars dealt with to be uniformly distributed throughout space, of some interest. We see that they are quite independent of the radius b over which we extend the sphere of our observations. Further, we remark that $\sigma_{m_1}{}^2$, $\sigma_{m_2}{}^2$, and $\sigma_{m_1 m_2}\, r_{m_1 m_2}$ ought in all cases where we may reasonably assume a uniform distribution to come out greater than $0{\cdot}524$.

(5) I will now proceed to modify these constants by supposing the stellar universe arranged in two spherical layers about the sun, each layer having uniform but individual density. Let δ_1 be the density up to radius b_1, and δ_2 the density from b_1 to b_2. Then it follows that

$$V_D = 0{\cdot}258\left\{1 + \frac{16\,(b_1 - b_2)^2\, b_1^3 b_2^3 \delta_2 (\delta_1 - \delta_2)}{(b_1^4(\delta_1 - \delta_2) + b_2^4\delta_2)^2}\right\}^{\frac{1}{2}},$$

$$V_\pi = 0{\cdot}577\left\{1 + \frac{4\,(b_1 - b_2)^2\, b_1 b_2 \delta_2 (\delta_1 - \delta_2)}{(b_1^2(\delta_1 - \delta_2) + b_2^2\delta_2)^2}\right\}^{\frac{1}{2}},$$

and $$25\,\sigma_P^2 = 0{\cdot}5240 + \frac{25\, b_1^3 b_2^3 \delta_2 (\delta_1 - \delta_2)(\log_{10} b_2/b_1)^2}{(b_1^3(\delta_1 - \delta_2) + b_2^3\delta_2)^2}.$$

It is clear that to raise the values of V_D, V_π, and $25\,\sigma_P^2$, we must make δ_2 *less than* δ_1. In other words, in a general way these parallax constants will

be higher if the stellar universe is rarer at greater distances from us. But the changes are not very rapid. If we make $b_2 = 2b_1$, then for $\delta_1 = 2\delta_2$, we find $25\sigma_P{}^2$ only 0·748, and for $\delta_1 = 9\delta_2$, a very improbable supposition, $25\sigma_P{}^2$ has only risen to 1·090. For $b_2 = 4b_1$, the values for the same densities are 0·661 and 1·419.

A dense nucleus round the sun appears the only way in which substantial change could be made in these parallax constants.

(6) I do not stay to discuss these points, because another method of investigating the problem will throw light on other matters also. If we supposed our stellar universe to be spheroidal in form, or that we dealt with a spheroidal portion of it, we should clearly be approaching the same result, after a certain limit, as supposing the density to decrease with the distance from the sun.

We need not reject such a system with the sun in the centre as necessarily idle, for we are only considering the nature of the changes which very rough approximations to the actual state of affairs will make in the parallax constants.

In particular, the limit to the oblate spheroid may be looked upon as a flat stellar universe, and this "bunlike" universe is worth consideration. Again, the limit to a prolate spheroid is a columnar universe, and, the constants of such a columnar universe being determined, we can superpose two such columnar universes, and obtain the parallax constants for a double drift system.

The polar axis of our spheroid being $2b$ and the equatorial axis $2a$, I find for the mean distance of the population from its centre

$$\bar{\mathrm{D}} = \tfrac{3}{8}b\left\{1+\frac{1+\eta^2}{\eta}\tan^{-1}\eta\right\} = \tfrac{3}{8}a\left\{\frac{1}{(1+\eta^2)^{\frac{1}{2}}}+\frac{(1+\eta^2)^{\frac{1}{2}}}{\eta}\tan^{-1}\eta\right\},$$

where $\eta = (a^2/b^2-1)^{\frac{1}{2}}$, for the oblate spheroid; and

$$\bar{\mathrm{D}} = \tfrac{3}{8}b\left\{1+\frac{1-\epsilon^2}{\epsilon}\log_e\sqrt{\frac{1+\epsilon}{1-\epsilon}}\right\},$$

where $\epsilon = (1-a^2/b^2)^{\frac{1}{2}}$, for the prolate spheroid. Further

$$\sigma_{\mathrm{D}} = \sqrt{\tfrac{1}{5}(2a^2+b^2)-\bar{\mathrm{D}}^2}$$

for both. Hence $\mathrm{V_D} = \sigma_{\mathrm{D}}/\bar{\mathrm{D}}$, can readily be found.

In the case of the parallax we have for the oblate spheroid

$$\bar{\varpi} = \frac{3}{2}\frac{c}{b}\frac{\tan^{-1}\eta}{\eta},$$

$$\mathrm{V}_\pi = \sqrt{\frac{4}{3}\frac{\sin\chi}{\chi^2}\log_e\cot\left(\frac{\pi}{4}-\frac{1}{2}\chi\right)-1},$$

where $\cos\chi = b/a$ and $\eta = \tan\chi$.

For the prolate spheroid

$$\bar{\pi} = \frac{3}{2}\frac{c}{b}\frac{1}{\epsilon}\log_e \sqrt{\frac{1+\epsilon}{1-\epsilon}},$$

$$V_\pi = \sqrt{\frac{4}{3}\frac{\psi\tan\psi}{\{\log_e\cot(\frac{1}{4}\pi-\frac{1}{2}\psi)\}^2}-1},$$

where $\cos\psi = a/b$ and $\epsilon = \sin\psi$.

These equations have been found by straightforward but somewhat laborious integration. The determinations of the mean, $\bar{P}$, and standard deviation, σ_P, of the parallax logarithm are not so simple.

We have

$$\tfrac{4}{3}\pi a^2 b\rho\bar{P}/\mu = 2\int_0^{\frac{1}{2}\pi}\int_0^r \rho\, 2\pi r\cos\theta\, r\, d\theta\, dr \log_e\frac{c}{r}$$

$$= 4\pi\int_0^{\frac{1}{2}\pi}\int_0^r \rho\log_e\frac{c}{r}\, d\left(\frac{1}{3}r^3\right) d\sin\theta,$$

$$= \frac{4\pi}{3}\int_0^{\frac{1}{2}\pi}\rho\left(r^3\log_e\frac{c}{r}+\frac{1}{3}r^3\right) d\sin\theta.$$

Let $r = av$, then $v^2 = (1+\eta^2\sin^2\theta)^{-1}$ for the oblate spheroid, and $\sin\theta = \eta^{-1}\sqrt{1/v^2-1}$. Change the integration to v, and we have

$$\bar{P} = \frac{\mu a}{b\eta}\int_1^{b/a}\left(\log_e\frac{c}{av}+\frac{1}{3}\right) d\sqrt{1-v^2} = \mu\left(\frac{1}{3}+\log_e\frac{c}{b}+\gamma\right),$$

on integrating by parts, where

$$\gamma = 1-\frac{\sqrt{1+\eta^2}}{\eta}\log_e(\eta+\sqrt{1+\eta^2}),$$

and vanishes when $\eta = 0$. Next,

$$\tfrac{4}{3}\pi a^2 b\rho\sigma_P^2 = 2\int_0^{\frac{1}{2}\pi}\int_0^r \rho\, 2\pi r\cos\theta\, r\, d\theta\, dr\left(\mu\log_e\frac{c}{r}-\bar{P}\right)^2,$$

$$= \tfrac{4}{3}\pi\rho\mu^2\int_0^{\frac{1}{2}\pi}\int_0^r d(\sin\theta)\, d(r^3)\left(\log_e\frac{c}{r}-\nu\right)^2,$$

where $\nu = \frac{1}{3}+\gamma$. Whence

$$\tfrac{4}{3}\pi a^2 b\rho\sigma_P^2 = \frac{4\pi\rho}{3}\mu^2\int_0^{\frac{1}{2}\pi}\int_0^r d(\sin\theta)\, d(r^3)\left(\log_e{}^2\frac{b}{r}-\nu^2\right)$$

$$= \frac{4\pi\rho}{3}\mu^2\int_0^{\frac{1}{2}\pi} d(\sin\theta)\left(r^3\log_e{}^2\frac{b}{r}+\frac{2r^3}{3}\log_e\frac{b}{r}+\tfrac{2}{9}r^3-\nu^2 r^3\right).$$

Put $r = av$ again, and change to an integration with regard to v. We find

$$\sigma_P^2 = \frac{a\mu^2}{b\eta}\int_1^{b/a}\left(\log_e{}^2\frac{b}{av}+\tfrac{2}{3}\log_e\frac{b}{av}+\tfrac{2}{9}-\nu^2\right) d\sqrt{1-v^2}.$$

Integrating by parts, the first term will be found to vanish between the limits, and we have

$$\sigma_P^2/\mu^2 = \frac{2a}{b\eta}\int_1^{b/a} \log_e \frac{b}{av}\,\frac{\sqrt{1-v^2}}{v}\,dv + \tfrac{2}{3}\gamma + \tfrac{2}{9} + (\tfrac{1}{3}+\gamma)^2,$$

or

$$\sigma_P^2/\mu^2 = \tfrac{1}{9} - \gamma^2 + I,$$

where I is the integral. It will be found that

$$I = -\gamma \log_e (1+\eta^2) - \frac{2a}{b\eta}\int_{b/a}^{1} \frac{\sqrt{1-v^2}}{v}\log_e \frac{1}{v}\,dv,$$

or

$$25\sigma_P^2/\mu^2 = 2{\cdot}778 - 25\gamma^2 - 25\gamma \log_e (1+\eta^2)$$
$$- \frac{50\sqrt{1+\eta^2}}{\eta}\int_{b/a}^{1} \frac{\sqrt{1-v^2}}{v}\log_e \frac{1}{v}\,dv.$$

To find the last integral we put

$$\int_{b/a}^{1} \frac{\sqrt{1-v^2}}{v}\log_e \frac{1}{v}\,dv = \int_{b/a}^{1} \frac{1}{v}\log_e \frac{1}{v}\,dv - \int_{b/a}^{1} \frac{1-\sqrt{1-v^2}}{v}\log_e \frac{1}{v}\,dv$$
$$= \tfrac{1}{2}\left(\log_e \frac{a}{b}\right)^2 - \int_{b/a}^{1} \frac{1-\sqrt{1-v^2}}{v}\log_e \frac{1}{v}\,dv.$$

The latter integral is finite from 0 to 1. The curve given by

$$z = \frac{1-\sqrt{1-v^2}}{v}\log_e \frac{1}{v}$$

was plotted for the range 0 to 1 and mechanically integrated.

For the prolate spheroid by a similar process we have

$$\bar{P} = \mu\left(\tfrac{1}{3} - \log_e \frac{c}{b} - \gamma'\right),$$

where

$$\gamma' = 1 - \frac{\sqrt{1-\epsilon^2}}{\epsilon}\sin^{-1}\epsilon,$$

and

$$\sigma_P^2/\mu^2 = \tfrac{1}{9} - \gamma'^2 - \gamma' \log_e (1-\epsilon^2) - \frac{2\sqrt{1-\epsilon^2}}{\epsilon}\int_1^{b/a} \frac{\sqrt{v^2-1}}{v}\log_e v\,dv,$$

or

$$25\sigma_P^2/\mu^2 = 2{\cdot}778 - 25\gamma'^2 - 25\gamma' \log_e (1-\epsilon^2) - \frac{50\sqrt{1-\epsilon^2}}{\epsilon}\int_1^{b/a} \frac{\sqrt{v^2-1}}{v}\log_e v\,dv.$$

To find the integral, I take

$$\int_1^{b/a} \frac{\sqrt{v^2-1}}{v}\log_e v\,dv = \int_1^{b/a} \log_e v\,dv - \int_1^{b/a} \frac{v-\sqrt{v^2-1}}{v}\log_e v\,dv$$
$$= \frac{b}{a}\left(\log_e \frac{b}{a} - 1 + \frac{a}{b}\right) - \int_1^{b/a} \frac{v-\sqrt{v^2-1}}{v}\log_e v\,dv.$$

The subject of the latter integral always remains finite. The curve $z = \dfrac{v-\sqrt{v^2-1}}{v}\log_e v$ was plotted and mechanically integrated up to the values required.

The following table gives the chief stages in the determination of $25\sigma_P^2$:—

Oblate Spheroid.

Ratio of axes.	γ.	$\frac{1}{2}\left(\log_e \frac{a}{b}\right)^2$.	$\int_{b/a}^{1} \frac{1-\sqrt{1-v^2}}{v} \log_e \frac{1}{v}\, dv$.
$a/b = 1000$	$-6\cdot6010$	$23\cdot85854$	$0\cdot13572$
$a/b = 100$	$-4\cdot2986$	$10\cdot60380$	$0\cdot13559$
$a/b = 10$	$-2\cdot0083$	$2\cdot65095$	$0\cdot12877$
$a/b = 3$	$-0\cdot8697$	$0\cdot60347$	$0\cdot09098$
$a/b = 2$	$-0\cdot5207$	$0\cdot24023$	$0\cdot05937$

Prolate Spheroid.

Ratio of axes.	γ'.	$\frac{b}{a}\left(\log_e \frac{b}{a} - 1 + \frac{a}{b}\right)$.	$\int_{1}^{b/a} \frac{v-\sqrt{v^2-1}}{v} \log_e v\, dv$.
$b/a = 2$	$+0\cdot3954$	$0\cdot38629$	$0\cdot08902$
$b/a = 3$	$+0\cdot5648$	$1\cdot29584$	$0\cdot16606$
$b/a = 10$	$+0\cdot8522$	$14\cdot02585$	$0\cdot35055$
$b/a = 100$	$+0\cdot9844$	$361\cdot51702$	$0\cdot48890$
$b/a = 1000$	$+0\cdot9984$	$5908\cdot75528$	$0\cdot59240$

Substituting in the above equations, we obtain, after some arithmetic, the values of V_D, V'_π, and $25\sigma_P^2$.

The following table results:—

	Ratio of axes.	V_π.	V_D.	$25\sigma_P^2$.
Oblate spheroid.	$a/b = \infty$	∞	$0\cdot391$	$1\cdot360$*
	$a/b = 1000$	$1\cdot771$	$0\cdot391$	$1\cdot360$
	$a/b = 100$	$1\cdot378$	$0\cdot391$	$1\cdot354$
	$a/b = 10$	$0\cdot914$	$0\cdot385$	$1\cdot210$
	$a/b = 3$	$0\cdot680$	$0\cdot343$	$0\cdot842$
	$a/b = 2$	$0\cdot622$	$0\cdot309$	$0\cdot680$
	$a/b = 1$	$0\cdot577$	$0\cdot258$	$0\cdot524$
Prolate spheroid.	$b/a = 2$	$0\cdot628$	$0\cdot369$	$0\cdot753$
	$b/a = 3$	$0\cdot703$	$0\cdot438$	$1\cdot104$
	$b/a = 10$	$1\cdot085$	$0\cdot606$	$2\cdot643$
	$b/a = 100$	$2\cdot532$	$0\cdot649$	$4\cdot657$
	$b/a = 1000$	$6\cdot046$	$0\cdot650$	$5\cdot148$
	$b/a = \infty$	∞	$0\cdot650$	$5\cdot239$†

* The limiting value of γ is $1-\log_e 2-\log_e \eta$, and again of

$$\sigma_P^2/\mu^2 = 2\cdot778 - 25\,(1-\log_e 2)^2 + 50\times 0\cdot13572.$$

† The limiting value of ϵ is 1 and of γ' is 1, and again of

$$\sigma_P^2/\mu^2 = 2\cdot778 - 25\,(1+\log_e(1-\epsilon^2)) - 50\left(\log_e \frac{b}{a} - 1\right) = 27\cdot778.$$

I shall speak of 0·524 as the "normal" value of the constant of the stellar system, $25\sigma_P^2$, since it is the value it would take if the stars were distributed at random through space. It is clear that to suppose the distribution spheroidal and not spherical will allow of some increase from the normal value, but far ess for the "bunlike" than the columnar universe. It is desirable at this stage to compare these numbers with actual data, however slender.

(7) Kapteyn's parallax determinations being excluded, I can only give results deduced from the 72 stars cited by Newcomb* as having the best determined parallaxes, and from 163 stars observed at Yale.† The zero and negative parallaxes must of necessity be excluded, because they lead to infinite distance D and infinite logarithm of parallax P. This reduced for V_D and $25\sigma_P^2$ the number of stars to 69 Newcomb and 122 Yale—admittedly most slender and imperfect material.

We have, after some considerable arithmetic,

	V_D.	V_π.	$25\sigma_P^2$.
Normal system	0·258	0·577	0·5240
Newcomb stars	1·075	0·924	4·212
Yale stars	1·017	1·443	3·111

For V_π I have, to see what effect it would produce, included the zero-parallax stars. The result is to separate more markedly the Newcomb and Yale series, and this was to be expected. For Newcomb excluded all but the best determined parallaxes, and so, many zero cases which would certainly be included by the Yale observers. For V_D the Newcomb and Yale stars give singularly accordant results, but wholly out of accord with the "normal" system. Nor is this to be wondered at. Parallax is subject to a very large error of observation, and our apparent σ_D and σ_π have not their true values, but values equal to $\sqrt{\sigma_D^2+\sigma_{E_1}^2}$ and $\sqrt{\sigma_\pi^2+\sigma_{E_2}^2}$, where σ_{E_1} and σ_{E_2} measure the variability in observational error in distance and parallax respectively. Now let us take $\sigma_{E_1} = \sigma_D$ and $\sigma_{E_2} = \sigma_\pi$, *i.e.* assume the variability in error to be as great as the quantity to be determined. This cannot be true, because to admit it is to assert that parallax determinations are wholly unreliable, but it will serve to indicate that the maximum limit to error does not suffice to reduce our observations to the "normal" values.

Dividing our apparent values by $\sqrt{2}$ we find $V_D = 0{\cdot}760$ and 0·719

* 'The Stars: a Study of the Universe,' 1902.

† Chase, Smith, and Elkins, "Parallax Observations on 163 Stars," 'Trans. Yale Univ. Observ.,' vol. 2, pp. 1—207.

V_π* $= 0{\cdot}653$ and $1{\cdot}020$ for the Newcomb and Yale stars respectively. The reduction is considerable, but it will not bring the observations into accord with the extreme values of V_D for our spheroidal universe, still less with the theoretical "normal" values of $V_D = 0{\cdot}258$, and $V_\pi = 0{\cdot}577$. The theoretical and observed values for $25\sigma_P^2$ are still further out of agreement.

It must be at once admitted that the argument against a uniform distribution of the stars, based on these data, is not of much weight unless as confirming other results. In the first place we must suppose that we take a random sample of all the stars within a given distance (b) of the sun, *i.e.* with greater than a given parallax. In both cases the stars have been selected because they had big proper motions. Newcomb's stars have been largely chosen because they had a parallax greater than 0·00, *i.e.* parallaxes only sensible in the third place of figures have been rejected, but we cannot assert that they are a random sample of all stars within a given sphere. Still the general agreement in V_D between the two independent series when we cut off all the zero (*i.e.* 0·00) parallaxes is remarkable, and carries some if not very great weight.

Returning now to our fundamental equation

$$\sigma_m^2 = \sigma_f^2 + 25\,\sigma_P^2,$$

we find for the 69 Newcomb stars $\sigma_m^2 = 6{\cdot}943$, which is clearly greater than the maximum possible for $25\sigma_P^2$, *i.e.* 4·212, and leaves σ_f^2 a minimum value of 2·631. But if we take the Yale stars we find $\sigma_m^2 = 2{\cdot}621$, which is *less* than the apparent value, 3·111, of $25\sigma_P^2$, so that if the latter were correct we should have a negative value for σ_f^2. This can only be satisfactorily accounted for on the assumption that f and P are not independent in our selection. In other words, although there is probably no organic relationship between intrinsic brightness and distance, still whenever we take a series of stars by any form of selection, we introduce a spurious correlation, which will generally of course be negative, between intrinsic brightness and parallax; that is to say, of the stars at a great distance—*i.e.* stars with small parallax—only those of great intrinsic brightness are admitted to our series.† The

* For both series together (245 stars) $V_\pi = 1{\cdot}361$, or with maximum correction 0·962, quite incompatible with 0·577.

† This question of a spurious correlation arising from the manner in which the observer selects parallax stars is not without considerable interest, and worth following up in the case of the Yale stars. Our general relation is $m = f - 5\,P$, which leads us at once to: $\sigma_f^2 = \sigma_m^2 + 10\sigma_m\sigma_P r_{mP} + 25\,\sigma_P^2$ and $\sigma_f\sigma_P r_{fP} = \sigma_m\sigma_P r_{mP} + 25\,\sigma_P^2$. Now we can directly find σ_m, σ_P, and r_{mP}. Omitting Algol we have for the 121 Yale stars with positive parallax: $\sigma_m = 1{\cdot}6190$, $\sigma_P = 0{\cdot}3540$, $r_{mP} = -0{\cdot}2179$, whence we deduce $\sigma_f = 2{\cdot}1225$ and $r_{fP} = 0{\cdot}6677$. Now (see p. 48), we have:

$$f = \text{a constant} - 2{\cdot}5\,\log_{10}F'$$

modification which this idea requires in our theory will be developed in the following sections.

(8) In dealing with star series it is not unusual to include all stars up to a given magnitude, and counts have been made which are fairly reliable of all stars up to the 7·5 magnitude; indeed, we may be said to have practically complete lists of all the lucid stars (< 6 mag.). It will be clear, then, that if we can modify our theory, so that we do not include all stars up to a given distance from the sun, but all stars up to a given magnitude, we shall be able to compare it much more satisfactorily with observation. Now, this process of cutting off from consideration all stars above a given magnitude introduces a spurious correlation between magnitude and parallax, or between logarithm of parallax and the light emission.

We reduce the "density" of the more distant layers of stars, because the fainter stars in these layers do not reach the minimum admissible magnitude.

In order to reach any conclusion, it is, I think, needful to assume a law of distribution of the size and intrinsic brightness factor of the stars.

Returning to our equation for the magnitude, we have (see p. 48), neglecting any absorption factor

$$\begin{aligned} m &= 10 + 2{\cdot}5 \log_{10} L_0 + 5 \log_{10} c - 2{\cdot}5 \log_{10} F - 5 \log_{10} \pi \\ &= 10 + 2{\cdot}5 \log_{10} L_0 - 2{\cdot}5 \log_{10} F + 5\mu \log_e D, \end{aligned}$$

where μ is again the modulus, and D is the distance. I shall put $\log_e D = x$, and instead of the f of the previous work write briefly y for

$$10 + 2{\cdot}5 \log_{10} L_0 - 2{\cdot}5 \log_{10} F.$$

Thus

$$m = y + 5\mu x,$$

and y will be a factor the variation of which depends solely on the size and intrinsic brightness of the individual star.*

Now, we need some law of frequency to distribute the character y, and it is not unreasonable to assume it in the first place to follow the same distribution—the Gaussian—which has been adopted by many astronomers for other stellar characters. I shall consider this point again; at present, it will be at least a guide to our theory, and indicate the possibility of

where F measures the amount of light emitted at the star. Hence if f and P are positively correlated, F and P will be negatively correlated, or we find in the Yale data that the light emitted and the parallax of the star are substantially correlated. This, of course, is an indirect result of the further stars being only included in the list when they have relatively greater absolute brilliancy than the near stars.

* x and y are used for the P and f of the earlier parts of this paper as more convenient for the following rather lengthy algebraic analysis.

extensions. We have now carefully to distinguish between the actual stars in the universe and those which will be found in our list, which we will suppose includes all, or at any rate a perfectly random sample of all, the stars up to magnitude m_0.

Let $\bar{y}_0$ be the mean value of y for all the actual stars, and suppose the frequency of y given by

$$z = \text{const.} \times e^{-\frac{1}{2}(y-\bar{y}_0)^2/\sigma_y^2} = \text{const.} \times e^{-\frac{1}{2}\eta^2/\sigma_y^2},$$

if we write $\eta = y - \bar{y}_0$ for brevity.

Assuming a uniform distribution of stars of density ρ, we have between D and D+δD

$$4\pi\rho \mathrm{D}^2\delta \mathrm{D} \text{ stars} = 4\pi\rho \mathrm{D}^3\delta \log_e \mathrm{D} = 4\pi\rho e^{3x}\,dx.$$

Hence, since the hypothesis is that the intrinsic brightness of the stars and their distance from the sun are independent, it follows that the number of stars having characters lying between y and $+\delta y$, x and $x+\delta x$ is of the form

$$\text{const.} \times e^{-\frac{1}{2}\eta^2/\sigma_y^2} e^{3x}\,dy\,dx = z_0 \times e^{-\frac{1}{2}\eta^2/\sigma_y^2+3x}\,dy\,dx, \text{ say.}$$

Now we are to take all possible values of x and y subject to the condition that $y+5\mu x$ does not exceed m_0. Clearly our limit is to be given by

$$m_0 - \bar{y}_0 = \eta + 5\mu x, \quad \text{or} \quad x = \frac{1}{5\mu}(\nu_0 - \eta),$$

where ν_0 is written for $m_0 - \bar{y}_0$ for brevity. Hence, if N be the total number of the selected stars,

$$\begin{aligned}
\mathrm{N} &= z_0 \int_{-\infty}^{+\infty}\int_{-\infty}^{\frac{1}{5\mu}(\nu_0-\eta)} e^{-\frac{1}{2}\eta^2/\sigma_y^2+3x}\,dx\,dy \\
&= \tfrac{1}{3} z_0 \int_{-\infty}^{+\infty} e^{-\frac{1}{2}\eta^2/\sigma_y^2+\frac{3}{5\mu}(\nu_0-\eta)}\,d\eta \\
&= \tfrac{1}{3} z_0 e^{\frac{3}{5\mu}\left(\nu_0+\frac{3}{10}\frac{\sigma_y^2}{\mu}\right)} \int_{-\infty}^{+\infty} e^{-\frac{1}{2}\left(\eta+\frac{3}{5}\frac{\sigma_y^2}{\mu}\right)/\sigma_y^2}\,d\eta \\
&= \tfrac{1}{3} z_0 e^{\frac{3}{5\mu}\left(\nu_0+\frac{3}{10}\frac{\sigma_y^2}{\mu}\right)} \sqrt{2\pi}\,\sigma_y.
\end{aligned}$$

This determines $z_0 = \dfrac{3\,\mathrm{N}}{\sqrt{2\pi}\,\sigma_y} e^{-\frac{3}{\mu}\left(\nu_0+\frac{3}{10}\frac{\sigma_y^2}{\mu}\right)}$ and the constant of our frequency distribution. We can now proceed to find the mean $\bar{\eta}$ and the standard deviation σ_η of the emitted lights of the selected population of stars.

We have

$$\mathrm{N}\bar{\eta} = z_0\int_{-\infty}^{+\infty}\int_{-\infty}^{\frac{1}{5\mu}(\nu_0-\eta)} \eta e^{-\frac{1}{2}\eta^2/\sigma_y{}^2+3x}\,dx\,d\eta$$

$$= \frac{z_0}{3}\int_{-\infty}^{+\infty} e^{-\frac{1}{2}\left(\eta+\frac{3\sigma_y{}^2}{5\mu}\right)^2/\sigma_y{}^2}\left(\eta+\frac{3\sigma_y{}^2}{5\mu}-\frac{3\sigma_y{}^2}{5\mu}\right)e^{\frac{3}{5\mu}\left(\nu_0+\frac{3}{10}\frac{\sigma_y{}^2}{\mu}\right)}\,d\eta\,,$$

$$= \frac{z_0}{3}\,e^{\frac{3}{5\mu}\left(\nu_0+\frac{3}{10}\frac{\sigma_y{}^2}{\mu}\right)}\left(-\frac{3\sigma_y{}^2}{5\mu}\right)\sqrt{2\pi}\,\sigma_y\,.$$

Therefore, $\bar{\eta} = -\dfrac{3}{5\mu}\sigma_y{}^2$, and $\bar{y} = \bar{y}_0+\eta = \bar{y}_0-\dfrac{3\sigma_y{}^2}{5\mu}$.

Similarly

$$\mathrm{N}\left(\sigma_\eta{}^2+\bar{\eta}^2\right) = z_0\int_{-\infty}^{+\infty}\int_{-\infty}^{\frac{1}{5\mu}(\nu_0-\eta)} \eta^2 e^{-\frac{1}{2}\eta^2/\sigma_y{}^2+3x}\,dx\,d\eta$$

$$= \tfrac{1}{3}z_0\int_{-\infty}^{+\infty} e^{-\frac{1}{2}\left(\eta+\frac{3\sigma_y{}^2}{5\mu}\right)^2/\sigma_y{}^2}\left(\eta+\frac{3\sigma_y{}^2}{5\mu}-\frac{3\sigma_y{}^2}{5\mu}\right)^2 e^{\frac{3}{5\mu}\left(\nu_0+\frac{3}{10}\frac{\sigma_y{}^2}{\mu}\right)}\,d\eta\,,$$

or

$$\sigma_\eta{}^2+\bar{\eta}^2 = \frac{1}{\sqrt{2\pi}\,\sigma_y}\left(\sqrt{2\pi}\,\sigma_y\,\sigma_y{}^2+\frac{9\sigma_y{}^4}{25\mu^2}\sqrt{2\pi}\,\sigma_y\right),$$

and accordingly $\sigma_\eta = \sigma_y$.

Thus, while the effect of cutting off the series at a given magnitude is to lower the mean light emitted, it makes no change whatever in the variability of light emitted. Neither the mean nor the standard deviation of the light emitted depends at all on the magnitude at which we close our series.

I now turn to the mean and variability of the logarithm of the distance.

$$\mathrm{N}\bar{x} = z_0\int_{-\infty}^{+\infty}\int_{-\infty}^{\frac{1}{5\mu}(\nu_0-\eta)} e^{-\frac{1}{2}\eta^2/\sigma_y{}^2}x\,e^{3x}\,dx\,dy$$

$$= \tfrac{1}{3}z_0\int_{-\infty}^{+\infty} e^{-\frac{1}{2}\eta^2/\sigma_y{}^2+\frac{3}{5\mu}(\nu_0-\eta)}\left(\frac{1}{5\mu}(\nu_0-\eta)-\frac{1}{3}\right)d\eta$$

$$= \tfrac{1}{3}z_0\int_{-\infty}^{+\infty} e^{-\frac{1}{2}\left(\eta+\frac{3}{5\mu}\sigma_y{}^2\right)^2/\sigma_y{}^2}\left[\frac{1}{5\mu}\nu_0-\tfrac{1}{3}+\frac{3\sigma_y{}^2}{25\mu^2}-\frac{1}{5\mu}\left(\eta+\frac{3}{5\mu}\sigma_y{}^2\right)\right]d\eta$$

$$\times\, e^{\frac{3}{5\mu}\left(\nu_0+\frac{3}{10}\frac{\sigma_y{}^2}{\mu}\right)}.$$

Hence

$$\bar{x} = \frac{1}{5\mu}\nu_0-\frac{1}{3}+\frac{3\sigma_y{}^2}{25\mu^2} = \frac{1}{5\mu}\left(m_0-\bar{y}_0+\frac{3\sigma_y{}^2}{5\mu}\right)-\frac{1}{3} = \frac{1}{5\mu}(m_0-\bar{y})-\frac{1}{3},$$

$$\therefore\qquad \mu\bar{x} = \frac{1}{\mathrm{N}}\,\mathrm{S}\,(\log_{10}\mathrm{D}) = \frac{1}{\mathrm{N}}\,\mathrm{S}\,(\log_{10}c-\log_{10}\pi) = \tfrac{1}{5}(m_0-\bar{y})-\frac{\mu}{3},$$

or

$$\bar{\mathrm{P}} = \frac{\mu}{3}+\log_{10}c-\tfrac{1}{5}(m_0-\bar{y}).$$

This should be compared with the result on p. 50. It gives the modified form taken by the mean logarithm of the parallax.

Further, since $m = y + 5\mu x$,

$$\begin{aligned}\bar{m} &= \bar{y} + 5\mu\bar{x}\\ &= \bar{y}_0 - \frac{3\sigma_y^2}{5\mu} + m_0 - \bar{y}_0 + \frac{3\sigma_y^2}{5\mu} - \frac{5\mu}{3}\\ &= m_0 - \frac{5\mu}{3} = m_0 - 0{\cdot}7238.\end{aligned}$$

Or, we have the remarkable general proposition that: *The mean of any random selection of stars not including any over the* m_0 *magnitude ought to be* 0·7238 *of a magnitude below this limiting value, if the stars are distributed uniformly through space.*

I next pass to the standard deviation of x. We have

$$\begin{aligned}\mathrm{N}(\sigma_x^2 + \bar{x}^2) &= z_0 \int_{-\infty}^{+\infty} \int_{-\infty}^{\frac{1}{5\mu}(\nu_0 - \eta)} e^{-\frac{1}{2}\eta^2/\sigma_y^2} x^2 e^{3x}\, dx\, dy\\ &= \tfrac{1}{3} z_0\, e^{\frac{3}{5\mu}\left(\nu_0 + \frac{3}{10}\frac{\sigma_y^2}{\mu}\right)} \int_{-\infty}^{+\infty} e^{-\frac{1}{2}\left(\eta + \frac{3}{5\mu}\sigma_y^2\right)^2 / \sigma_y^2} \left[\frac{1}{25\mu^2}(\nu_0 - \eta)^2 \right.\\ &\qquad\qquad \left. - \frac{2}{3}\frac{1}{5\mu}(\nu_0 - \eta) + \tfrac{2}{9}\right] d\eta.\end{aligned}$$

Whence, after some reduction, we find

$$\sigma_x^2 = \frac{1}{9} + \frac{1}{25\mu^2}\sigma_y^2.$$

But it is easy to see that $\sigma_{\log_e \mathrm{D}}$ is identical with $\sigma_{\log_e \pi}$, or σ_{P}/μ of our previous notation. Thus

$$25\sigma_{\mathrm{P}}^2 = \tfrac{25}{9}\mu^2 + \sigma_y^2 = 0{\cdot}5240 + \sigma_y^2.$$

We see accordingly that the quantity $25\sigma_{\mathrm{P}}^2$ when we cut off at a given magnitude will for every magnitude be still constant, but it will exceed by the quantity σ_y^2 the "normal" value on the assumption that a random sample has been taken of all stars.

I now proceed to find the product moment of x and y with a view to determining their correlation.

$$\begin{aligned}\mathrm{N}(\bar{p}_{xy} + \bar{\eta}\bar{x}) &= z_0 \int_{-\infty}^{+\infty} \int_{-\infty}^{\frac{1}{5\mu}(\nu_0 - \eta)} e^{-\frac{1}{2}\eta^2/\sigma_y^2 + 3x} \eta x\, dx\, dy\\ &= \tfrac{1}{3} z_0 \int_{-\infty}^{+\infty} e^{-\frac{1}{2}\left(\eta + \frac{3}{5\mu}\sigma_y^2\right)^2 / \sigma_y^2}\, \eta \left(\frac{1}{5\mu}(\nu_0 - \eta) - \tfrac{1}{3}\right) d\eta\, e^{\frac{3}{5\mu}\left(\nu_0 + \frac{3}{10\mu}\sigma_y^2\right)}.\end{aligned}$$

Whence, after some reductions, and substituting for $\bar{x}$ and $\bar{y}$, we find:

$$\bar{p}_{xy} = -\frac{1}{5\mu}\sigma_y^2.$$

Reverting to our original equation,

$$m = y + 5\mu x,$$

we have

$$\overline{m} = \overline{y} + 5\mu\overline{x};$$

$\therefore$

$$m - \overline{m} = y - \overline{y} + 5\mu(x - \overline{x})$$

and

$$\sigma_m^2 = \sigma_\eta^2 + 10\mu\overline{p}_{x\eta} + 25\mu^2\sigma_x^2$$

$$= \sigma_y^2 - 2\sigma_y^2 + \frac{25\mu^2}{9} + \sigma_y^2$$

$$= \frac{25\mu^2}{9} = 0{\cdot}5240.$$

Thus the effect of cutting off our sample at a given magnitude produces a most important result; we replace the statement that σ_m^2 must always be greater than 0·5240 by a much more striking one: *The standard deviation of the magnitudes of a series of stars cut off at a given magnitude is constant, whatever that magnitude may be, and equal to the normal value of the* $25\,\sigma_{\mathrm{P}}^2$ *parallax constant.*

We have here a very simple test of the uniform distribution of the stars through space.

I return to the result

$$\overline{p}_{xy} = -\frac{1}{5\mu}\sigma_y^2.$$

But

$$r_{xy} = \overline{p}_{xy}/\sigma_y\sigma_x = -\frac{1}{5\mu}\frac{\sigma_y}{\sigma_x} = \frac{-\sigma_y}{5\sigma_{\mathrm{P}}} = -\frac{1}{\sqrt{1+0{\cdot}524/\sigma_y^2}}.$$

There is thus always a *negative* correlation between x and y, which again means a positive correlation between the logarithm of the parallax and y, or, finally, a negative correlation between the logarithm of the parallax (or the parallax) and light actually emitted. In other words, cutting off at a given magnitude has introduced a spurious correlation between parallax and light emitted, between a star's distance from the sun and its size and intrinsic brightness. It is not probable that such a relationship actually exists in the totality of the stellar universe, but it should be apparently demonstrable in all actual collections of stars. For example, suppose the maximum light to be emitted by stars belonging to certain spectral (or colour) classes, then we should expect to find a correlation between parallax and spectral class of the above character. I think this "cutting off at a given magnitude" is most probably the source of the fairly high correlation (0·36) already noted by me as existing for the Yale observations between spectral class and parallax.*

* 'Monthly Notices, R.A.S.,' vol. 68, p. 428, 1908.

Further, this cutting off at a given magnitude will also modify the natural relationship between magnitude and parallax. Taking

$$m-\bar{m} = y-\bar{y}+5\mu(x-\bar{x})$$

and multiplying by $(x-\bar{x})$, summing and dividing by the number of pairs in the usual way, we have

$$\bar{p}_{mx} = \sigma_m\sigma_x r_{xm} = \bar{p}_{xy}+5\mu\sigma_x^2$$
$$= -\frac{1}{5\mu}\sigma_y^2+\frac{5\mu}{9}+\frac{1}{5\mu}\sigma_y^2 = \frac{5\mu}{9}.$$

Thus, $\bar{p}_{mx} = 0{\cdot}2413$, and is a constant.

Further, since $\sigma_m = \frac{5}{3}\mu$, we have

$$r_{xm} = \frac{1}{3}\frac{1}{\sigma_x} = \frac{1}{\sqrt{1+\frac{9}{25\mu^2}\sigma_y^2}} = \frac{1}{\sqrt{1+\frac{\sigma_y^2}{0{\cdot}5240}}}.$$

Finally, the correlation of the logarithm of the parallax with the magnitude will be given by

$$r_{Pm} = -\frac{1}{\sqrt{1+\frac{\sigma_y^2}{0{\cdot}5240}}}.$$

From the value previously given for $r_{xy} = -r_{Py}$ we find:

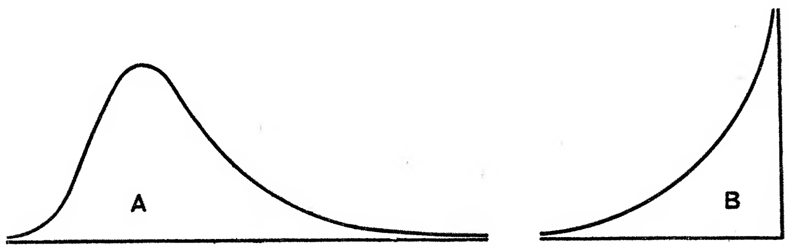
$$r_{Py}^2 = 1-r_{Pm}^2,$$

or we are able from observable quantities to determine the correlation of parallax with the actual light emitted.* It may be noted that when there is a high correlation between parallax and magnitude, a low correlation will be found between parallax and actual light emitted.

(9) Before we proceed to compare the results of the preceding section with observation, it is fitting to consider how far they really involve the assumption made that the lights actually emitted by the stars obey a Gaussian frequency distribution. It is probably quite as legitimate to assume this as to suppose that stellar proper-motions follow the Gaussian law, but I wish to show that it is by no means needful to assume this in order to obtain the chief results reached in the previous section. I have taken the Gaussian curve in the first place because it has a certain sanction from astronomical usage, but our results are really much more general. It is open to anyone to argue that the universe is a dying system, that there are a majority of dark stars, or stars of small intrinsic brilliancy. If this be true, not only will the distri-

* In the case of the Yale stars $\sigma_y^2 = \sigma_f^2 = 4{\cdot}5051$ and $\sigma_y^2/0{\cdot}5240 = 8{\cdot}5975$, and we find $r_{Pm} = -0{\cdot}3228$ and $r_{Pf} = 0{\cdot}9464$, against the observed values of $-0{\cdot}2179$ and $0{\cdot}6677$. This is, of course, further evidence that the Yale series is not a random sample "up to a given magnitude" of stars distributed uniformly through space.

bution of intrinsic brilliancies give a skew frequency curve, but the distribution of the logarithms of the factors which measure actually emitted lights will do the same thing. Accordingly we need, to cover this case, a frequency distribution far more general than the Gaussian, and one that can represent all degrees of skewness, from a type like A to one like B.



This is met by using the curve

$$z = z_0 \left(1-\frac{\eta}{b}\right)^p e^{\gamma\eta},$$

which ranges from $\eta = b$ to $\eta = -\infty$, and which can take the form B if p be negative.

I have discussed this curve at length in a memoir in the 'Philosophical Transactions,' vol. 186, A, p. 373.

By supposing y or η to have this type of distribution, we shall at any rate determine how far our results are peculiar to the form or symmetry of a Gaussian frequency.

The form of the frequency surface will now be

$$z = z_0 \left(1-\frac{\eta}{b}\right)^p e^{\gamma\eta+3x},$$

and we have at once

$$\begin{aligned} \mathrm{N} &= z_0 \int_{-\infty}^{b} \int_{-\infty}^{\frac{1}{5\mu}(\nu_0-\eta)} \left(1-\frac{\eta}{b}\right)^p e^{\gamma\eta+3x}\, dx\, d\eta \\ &= \tfrac{1}{3} z_0 \int_{-\infty}^{b} \left(1-\frac{\eta}{b}\right)^p e^{\left(\gamma-\frac{3}{5\mu}\right)\eta+\frac{3\nu_0}{5\mu}}\, d\eta. \end{aligned}$$

Assume $\zeta = \left(\gamma-\dfrac{3}{5\mu}\right)(b-\eta) = \lambda\left(1-\dfrac{\eta}{b}\right)$, so that $\lambda = b\left(\gamma-\dfrac{3}{5\mu}\right)$, and we have

$$\mathrm{N} = \tfrac{1}{3} z_0 e^{\frac{3\nu_0}{5\mu}+\lambda} \frac{b}{\lambda^{p+1}} \int_0^\infty \zeta^p e^{-\zeta}\, d\zeta = \tfrac{1}{3} z_0 \frac{e^{b\gamma+\frac{3}{5\mu}(\nu_0-b)}\, b}{\lambda^{p+1}} \Gamma(p+1).$$

Thus
$$z_0 = \frac{3\,\mathrm{N}\lambda^{p+1}}{b\Gamma(p+1)} e^{-b\gamma-\frac{3}{5\mu}(\nu_0-b)}.$$

Next:

$$\mathrm{N}\bar{\eta} = z_0 \int_{-\infty}^{b} \int_{-\infty}^{\frac{1}{5\mu}(\nu_0-\eta)} \left(1-\frac{\eta}{b}\right)^p e^{\gamma\eta+3x}\, \eta\, dx\, d\eta$$

$$= \tfrac{1}{3} z_0 \frac{e^{b\gamma+\frac{3}{5\mu}(\nu_0-b)}\, b}{\lambda^{p+1}} \int_0^\infty \zeta^p e^{-\zeta}\, b \left(1-\frac{\zeta}{\lambda}\right) d\zeta,$$

$$\bar{\eta} = \frac{b}{\Gamma(p+1)} \int_0^\infty \zeta^p e^{-\zeta} \left(1-\frac{\zeta}{\lambda}\right) \delta\zeta$$

$$= b\left(1-\frac{p+1}{\lambda}\right).$$

Now η is measured from the mean value, and therefore it is needful that* $p+1 = b\gamma$. Hence

$$\bar{\eta} = b\left(1-\frac{b\gamma}{\lambda}\right) = -\frac{3}{5\mu}\frac{b^2}{\lambda} = -\frac{3}{5\mu}\frac{p+1}{\gamma^2}\frac{1}{1-\frac{3}{5\mu\gamma}}.$$

Next

$$\mathrm{N}(\sigma_\eta^2+\bar{\eta}^2) = z_0 \int_{-\infty}^{b} \int_{-\infty}^{\frac{1}{5\mu}(\nu_0-\eta)} \left(1-\frac{\eta}{b}\right)^p e^{\gamma\eta+3x}\, \eta^2\, dx\, d\eta$$

$$= \frac{\tfrac{1}{3} z_0 e^{b\gamma+\frac{3}{5\mu}(\nu_0-b)} b}{\lambda^{p+1}} \int_0^\infty \zeta^p\, b^2 \left(1-\frac{2\zeta}{\lambda}+\frac{\zeta^2}{\lambda^2}\right) e^{-\zeta}\, d\zeta$$

Hence $$\sigma_\eta^2+\bar{\eta}^2 = b^2\left(1-\frac{2}{\lambda}(p+1)+\frac{(p+2)(p+1)}{\lambda^2}\right)$$

and $$\sigma_\eta^2 = \frac{b^2(p+1)}{\lambda^2} = \frac{p+1}{\gamma^2}\left(1-\frac{3}{5\mu\gamma}\right)^{-2}$$

$$= \sigma_y^2\left(1-\frac{3}{5\mu\gamma}\right)^{-2}, \dagger$$

that is $$\sigma_\eta = \sigma_y \Big/ \left(1-\frac{3}{5\mu\gamma}\right).$$

Further, we may write

$$\bar{\eta} = -\frac{3}{5\mu}\sigma_y^2 \Big/ \left(1-\frac{3}{5\mu\gamma}\right).$$

Thus σ_η and $\bar{\eta}$ differ from the values on the Gaussian hypothesis, *i.e.* σ_y and $-\frac{3}{5\mu}\sigma_y^2$, by the introduction of the factor $1/\left(1-\frac{3}{5\mu\gamma}\right)$, which becomes unity when $\gamma = \infty$. Still the two general propositions remain true, *i.e.* that

* 'Phil. Trans.,' A, vol. 186, p. 374.

† $\sigma_y^2 = (p+1)/\gamma^2$; see *loc. cit.*, p. 373.

when we cut off our series at a given magnitude, the mean of the actual lights emitted is reduced by a constant quantity, and the variability of these lights is constant, whatever be that magnitude.

Turning next to the mean and variability of the logarithm of the distance, we find

$$\mathrm{N}\bar{x} = z_0 \int_{-\infty}^{b} \int_{-\infty}^{\frac{1}{5\mu}(\nu_0-\eta)} \left(1-\frac{\eta}{b}\right)^p e^{\gamma\eta+3x}\, x\, dx\, d\eta$$

$$= \frac{\frac{1}{3}z_0 e^{b\gamma+\frac{3}{5\mu}(\nu_0-b)}b}{\lambda^{p+1}} \int_0^\infty \zeta^p \left(\frac{\nu_0}{5\mu}-\frac{1}{3}-\frac{b}{5\mu}+\frac{b}{5\mu}\frac{\zeta}{\lambda}\right) e^{-\zeta} d\zeta,$$

Hence
$$\bar{x} = \frac{\nu_0}{5\mu}-\frac{1}{3}-\frac{b}{5\mu}+\frac{b}{5\mu\lambda}(p+1),$$

$$= \frac{\nu_0}{5\mu}-\frac{1}{3}+\frac{3}{25\mu^2}\,\frac{b}{\gamma-\dfrac{3}{5\mu}},$$

$$= \frac{\nu_0}{5\mu}-\frac{1}{3}+\frac{3}{25\mu^2}\,\frac{\sigma_y^2}{1-\dfrac{3}{5\mu\gamma}},$$

which differs from the Gaussian frequency result solely by the presence of the factor $1-\dfrac{3}{5\mu\gamma}$ in the denominator of the last term.

Clearly we can write

$$\bar{x} = \frac{1}{5\mu}(m_0-y_0)-\frac{1}{3}-\frac{1}{5\mu}\bar{\eta},$$

leading, as on p. 59, to the result

$$\bar{\mathrm{P}} = \tfrac{1}{3}\mu+\log_{10}c-\tfrac{1}{5}(m_0-\bar{y}).$$

Now $\bar{y}$ is a constant whatever be m_0, for it equals $y_0-\bar{\eta}$. Hence the mean logarithm of the parallax, when we increase the magnitude at which we terminate our series, decreases by $\frac{1}{5}$ (difference of cutting off magnitudes).

Proceeding next to the value of σ_x, we have

$$\mathrm{N}(\sigma_x^2+\bar{x}^2) = z_0 \int_{-\infty}^{b} \int_{-\infty}^{\frac{1}{5\mu}(\nu_0-\eta)} \left(1-\frac{\eta}{b}\right)^p e^{\gamma\eta+3x} x^2\, dx\, d\eta$$

$$= \tfrac{1}{3}z_0 \frac{e^{b\gamma+\frac{3}{5}(\nu_0-b)}b}{\lambda^{p+1}} \int_0^\infty \zeta^p \left(\frac{1}{25\mu^2}(\nu_0-\eta)^2-\frac{2}{3}\,\frac{1}{5\mu}(\nu_0-\eta)+\frac{2}{9}\right) e^{-\zeta}\, d\zeta.$$

Substituting for η in terms of ζ, integrating out and using the above value of $\bar{x}$, we find after some rather lengthy reductions that

$$\sigma_x^2 = \frac{1}{9}+\frac{1}{25\mu^2}\sigma_y^2\left(1-\frac{3}{5\mu\lambda}\right)^{-2} = \frac{1}{9}+\frac{1}{25\mu^2}\sigma_\eta^2.$$

We have then, as on p. 60,

$$25\sigma_{\mathrm{P}}^2 = \frac{25}{9}\mu^2+\sigma_\eta^2 = 0\cdot5240+\sigma_\eta^2 = 0\cdot5240+\sigma_y^2\left(1-\frac{3}{5\mu\gamma}\right)^{-2}.$$

Thus, as before, $25\sigma_{\mathrm{P}}^2$ is a constant quite independent of m_0, but exceeding its "normal" value by a fixed amount.

We may add here the value of the mean magnitude. As on p. 60

$$\begin{aligned}\overline{m} &= \overline{y}+5\mu\overline{x}\\ &= y_0-\frac{3}{5\mu}\sigma_y^2\left(1-\frac{3}{5\mu\gamma}\right)^{-1}+\nu_0-\frac{5\mu}{3}+\frac{3}{5\mu}\sigma_y^2\left(1-\frac{3}{5\mu\lambda}\right)^{-1}\\ &= m_0-\frac{5\mu}{3} = m_0-0\cdot7238.\end{aligned}$$

Thus our general proposition of p. 60, that the mean magnitude of a system of stars up to a given magnitude is always less by 0·7238 than that magnitude, is established on the broader basis of non-Gaussian variability.

Lastly, I take the product moment of x and η

$$\begin{aligned}\mathrm{N}(\overline{p}_{xy}+\overline{x}\overline{\eta}) &= z_0\int_{-\infty}^{b}\int_{-\infty}^{\frac{1}{5\mu}(\nu_0-\eta)}\left(1-\frac{\eta}{b}\right)^p e^{\gamma\eta+3x}\,\eta\, x dx\, d\eta\\ &= \tfrac{1}{3}z_0\frac{e^{\gamma b+\frac{3}{5\mu}(\nu_0-b)}b^2}{\lambda^{p+1}}\int_0^\infty \zeta^p\left(1-\frac{\zeta}{\lambda}\right)\left\{\frac{1}{5\mu}\left(\nu_0-b+\frac{b\zeta}{\lambda}\right)-\frac{1}{3}\right\}e^{-\zeta}d\zeta,\end{aligned}$$

whence, integrating on the same lines as before, by aid of the Γ-function, we find

$$\overline{p}_{x\eta} = -\frac{b^2}{5\mu\lambda^2}(p+1) = -\frac{1}{5\mu}\sigma_\eta^2 = -\frac{1}{5\mu}\frac{\sigma_y^2}{\left(1-\frac{3}{5\mu\gamma}\right)^2}.$$

But

$$\begin{aligned}\sigma_m^2 &= \sigma_\eta^2+10\mu\overline{p}_{x\eta}+25\mu^2\sigma_x^2\\ &= \sigma_\eta^2-2\sigma_\eta^2+\frac{25\mu^2}{9}+\sigma_\eta^2\\ &= \frac{25\mu^2}{9} = 0\cdot5240.\end{aligned}$$

Thus, precisely as before with the Gaussian law, we have obtained with this more general frequency the same result, namely, whatever be the magnitude at which we cut off, the standard deviation of the magnitudes of the series is constant and equal to the "normal" value of $25\sigma_{\mathrm{P}}^2$ for an unselected universe.

We may conclude with the correlations

$$r_{xy} = -\left(1+\frac{0\cdot5240}{\sigma_\eta^2}\right)^{-\frac{1}{2}} = -r_{\mathrm{P}y},$$

$$r_{xm} = \left(1+\frac{\sigma_\eta^2}{0\cdot5240}\right)^{-\frac{1}{2}} = -r_{\mathrm{P}m},$$

and

$$r_{\mathrm{P}y}^2 = 1-r_{\mathrm{P}m}^2.$$

It follows therefore that all the results previously obtained for a Gaussian distribution hold for the much more general type of frequency now adopted. The only differences depend upon slight changes involving the constant factor $\left(1-\frac{3}{5\mu\gamma}\right)$.

The distribution I have selected for the emitted lights enables us to suppose brilliant stars or even dark stars to be in a majority. It is sufficiently wide, I think, to cover any probable distribution of the intrinsic brilliancies of the stellar universe very closely. The main conclusions, however, that

$$\overline{m} = m_0 - 0\cdot7238, \qquad \sigma_m{}^2 = 0\cdot5240,$$

I suspect to be capable of simple proofs independent of any assumption as to frequency distribution. I have not stayed to seek for such proofs, but given those by which I have myself reached these results. They provide the mean magnitude and magnitude variability of any catalogue or random sample of stars up to a given magnitude, for example, of all the stars which could be seen with a telescope of given aperture—of course, on the assumption that the stars are distributed at random through space. They are thus fundamental in determining whether such a distribution is the actual one or not. There are two points to be borne in mind here. First our discussion is perfectly independent of direction, we have only supposed that spherical shells about the sun as centre, if of equal volume, contain the same number of stars. These stars may or may not be clustered in individual shells towards one or more directions. Our propositions apply to the frequency at each distance, and have no relation to direction.

Secondly, we have seen that

$$\sigma_m{}^2 = 0\cdot5240 = 25\,\sigma_{\mathrm{P}}{}^2,$$

or takes the value that $25\sigma_{\mathrm{P}}{}^2$ does for a uniform spherical stellar universe. I strongly suspect that this principle is general, namely, that the value of $\sigma_m{}^2$ when we cut off at a given magnitude is equal to the value of the parallax constant, $25\sigma_{\mathrm{P}}{}^2$, for the total universe, whatever be its assumed form. If this be true, then the values given in the table on p. 54 will have additional meaning. They show the modifications possible in $\sigma_m{}^2$ when we suppose the stellar universe spheroidal rather than spherical in form—still maintaining a uniform distribution of stars. As I have indicated, an ellipsoidal rather than a spherical form for the stellar universe appears to raise the value of $25\sigma_{\mathrm{P}}{}^2$ and accords more closely, as we shall see, with experience. I hope later to discuss how far $\sigma_m{}^2 = 25\sigma_{\mathrm{P}}{}^2$ is either approximately or absolutely true for other than spherical stellar distributions.

(10) It remains now to consider how far the results reached in the

previous sections, (8) and (9), are consistent with observation. Luckily we possess very accurate counts of the stars up to magnitude 7·25. The best of these is probably that due to Prof. Pickering,* and I shall make use of that here. Even if in some of the categories the counts are not absolutely accurate, the divergencies will occur in the higher magnitudes, where they will, owing to the large numbers dealt with, make no impression on the values of the means or standard deviations.

The table below gives the frequency up to each magnitude in the second column. In the third and fourth columns are given the actual and theoretical mean magnitudes up to the magnitude in the first column, the fifth and sixth columns give the square of the actual and theoretical standard deviations. The seventh and eighth columns give the differences of the observed and calculated values registered in the third to sixth columns. The means and standard deviations were calculated by the summation method discussed by Palin Elderton.†

1.	2.	3.	4.	5.	6.	7.	8.
Up to magnitude.	Observed frequency.	Mean.		Sq. standard deviation.		Differences.	
		Observed.	Theoretical.	Observed.	Theoretical.	Δm.	$\Delta\sigma_m^2$.
0·25	5	—	—	—	—	—	—
0·75	10	—	—	—	—	—	—
1·25	18	—	—	—	—	—	—
1·75	32	—	—	—	—	—	—
2·25	58	1·440	1·526	0·423	0·524	−0·086	−0·101
2·75	105	1·914	2·026	0·512	0·524	−0·112	−0·012
3·25	193	2·409	2·526	0·571	0·524	−0·117	+0·047
3·75	336	2·874	3·026	0·619	0·524	−0·152	+0·095
4·25	589	3·357	3·526	0·664	0·524	−0·169	+0·140
4·75	1067	3·869	4·026	0·689	0·524	−0·157	+0·165
5·25	1972	4·388	4·526	0·691	0·524	−0·138	+0·167
5·75	3562	4·885	5·026	0·688	0·524	−0·141	+0·164
6·25	6284	5·368	5·526	0·695	0·524	−0·158	+0·171
6·75	11004	5·853	6·026	0·711	0·524	−0·173	+0·187
7·25	17955	6·297	6·526	0·748	0·524	−0·229	+0·224

It will be obvious from this table that while our formulæ $\sigma_m^2 = 0·524$ and $\bar{m} = m_0 - 0·724$, express approximately the truth (never differing by a quarter of a magnitude from it), yet the observed values differ sensibly and almost regularly from the calculated values—the higher the limiting magnitude the greater is the divergence. That is to say, the more stars we take into our consideration,

* "Distribution of the Stars," 'Annals of Harvard College Observatory,' vol. 48, pp. 149—185.

† 'Biometrika,' vol. 4, pp. 374—8.

the less accurate appears the assumption of the uniform distribution of the stars through space. I think we are fully justified on the basis of this table in stating that a random distribution of the stars through a spherical space is inconsistent with observed facts. Or, if such a random distribution exists, then there must be a correlation between distance and intrinsic brilliancy. Such an apparent correlation would be produced, of course, if light were to any extent absorbed in its transmission through space. The few stars of lower than the third magnitude may be reasonably supposed uniformly scattered through space, since they fairly closely satisfy the necessary conditions; beyond this magnitude we have *continuous* deviations which cannot possibly be looked upon as the random or irregular variations from a chance scattering.

(11) It may hardly be needful to bring further evidence that the stellar universe is not a random distribution through space of stars of varying size and intrinsic brilliancy. But it may still be of interest to illustrate one or two points from different material. I chose Burnham's 'Catalogue of Double Stars' because the selection there is of a somewhat different character to a star count, and further because there are records of stars up to a much higher magnitude. Confining our attention to primaries only, we have the following table:—*

Range of magnitude.	Number of stars.	$\overline{m}$.*	σ^2_m.
Up to 6·2	813	5·204 (5·496)	1·008
„ 8·2	4947	7·112 (7·496)	1·087
„ 10·2	9711	7·979 (9·496)	1·385
„ 12·2	9867	8·025 (11·496)	1·517
„ 15·5	9900	8·042 (14·796)	1·596

* Theoretical means are placed in brackets; of course, no stress must be laid on more than the first two or three entries, and the whole evidence is not comparable with that from total counts.

It is clear that σ_m^2 here starts a good deal higher than the star counts leave off. But the general tendency is the same, the mean does not increase fast enough, and the variability, instead of remaining constant, tends to rapidly increase. Very faint doubles may easily be overlooked, but, I think, this is far from explaining the *whole* evidence of the above table, which is in the same direction as we have seen in the total counts of lower magnitudes. Double stars even more markedly than the whole stellar system do not seem to obey the condition of uniform distribution through space.

Thus from several directions we find evidence accumulating that the stars cannot, without regard to their size or intrinsic brilliancy, be considered as on

* I owe this table to the kindness of Mr. P. F. Everitt.

the average uniformly distributed through space. The only way I can see to save such an hypothesis is either to assume a limited universe, not spherical in form, or to suppose light absorbed in transit. It will be interesting to inquire further into the effect of modifying the form of a uniformly dense stellar universe, and secondly to inquire whether when we confine our stars to one spectral class we get any difference in the general nature of the above deviations.

In conclusion, I must acknowledge my very great indebtedness to Miss Julia Bell, M.A., for much aid in the laborious calculations, of which only the final results are tabled in this paper; to Mr. E. Cunningham for a suggestion as to the best form in which to put the integral required in the case of the oblate spheroid universe; and to Mr. C. H. Williams for mechanically integrating and tabling both oblate and prolate spheroid integrals.

The Liberation of Helium from Minerals by the Action of Heat.

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Introductory.

It is well known that the helium contained in radio-active minerals can be liberated by the action of heat, but, hitherto, no exact quantitative experiments have been made on the subject, and it was thought desirable to conduct such experiments with a view:—

(1) To ascertain whether heat may not be used as a substitute for chemical methods in the case of minerals and rocks for which those methods are not convenient,* and, if so, the temperatures necessary.

(2) To throw, if possible, more light on the way in which the gas is retained within the mineral.

A number of experiments have been made by Travers† on cleveite and

* Chemical methods break down (*a*) when the helium-content is very small, so that large quantities have to be treated to liberate a measurable volume of gas; (*b*) when no simple reaction is available, or when such a reaction gives rise to large volumes of other gases. Iron sulphide, which occurs in many geological horizons, is an example of this.

† Travers, 'Roy. Soc. Proc.,' A, vol. 64, p. 140. The minerals were heated to redness in a hard glass tube, and, therefore, to a temperature not exceeding 600° C.